
EVERYTHING
INTERESTING
HAPPENS
ON A
CANTOR SET

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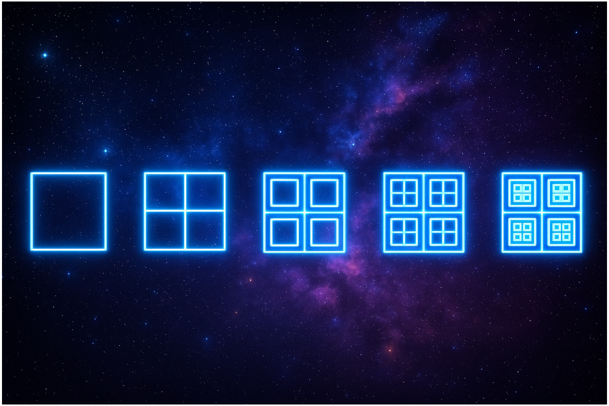

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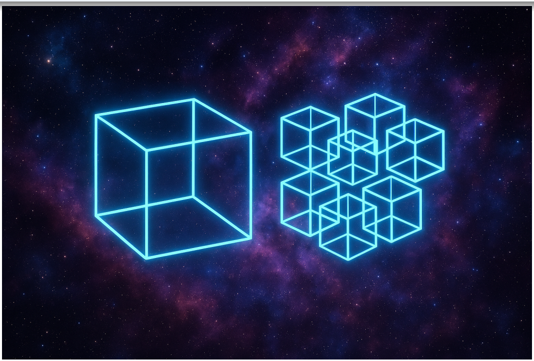

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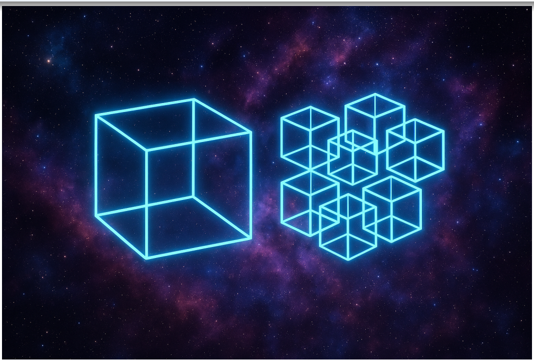

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The shift is $\frac{1}{2} r_k a_{k+1}$. Typical case $r_{k+1} = \frac{1}{2} r_k \beta$ for some $\beta \in (0,1)$.

Perfect self similarity: $z_{k+1}(a) = z_k(a) + \frac{1}{2} r_k = \sum_{j=0}^{k} \frac{1}{2} \beta^j a_j$

The Cantor set is closed and totally disconnected.

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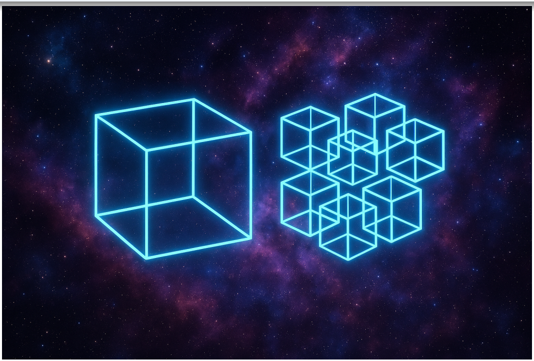

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Cantor sets are the stairway to heaven

Choose $\beta = 2^{\frac{p}{n-p}}$, then the Cantor set has positive finite $\mathcal{H}^{n-p}$.

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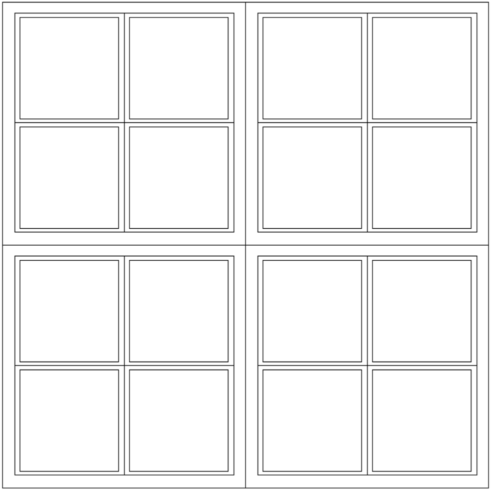
The Cantor set has zero p-capacity!

Cantor sets may have positive measure



Any Jordan curve of positive measure must contain a Cantor set of positive measure.

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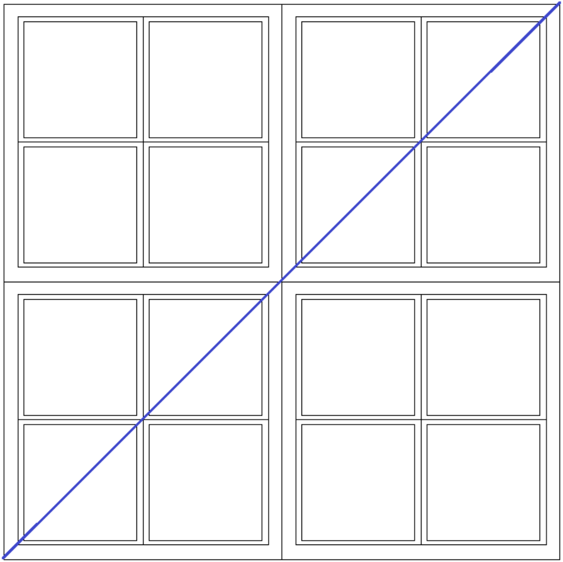

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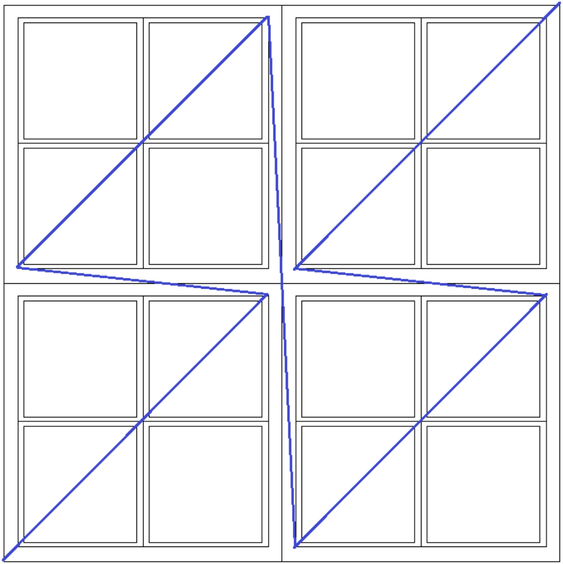

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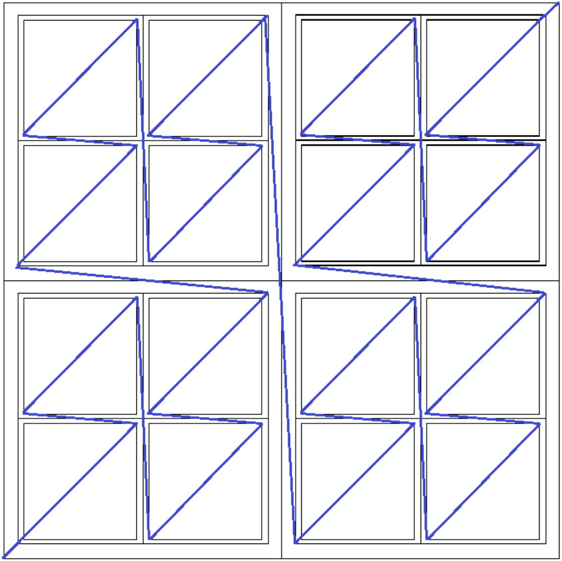
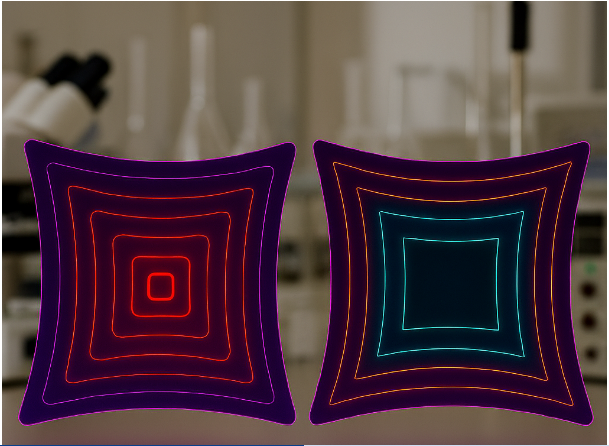


Between Cantor sets

Construct two cantor sets and map each frame onto each frame by a square-radial map.

(Or use a Cartesian product of such maps.)



A typical trick - big-small-big

Take a Cantor set of positive measure and squeeze it into one of zero measure.

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Do something weird to the small Cantor set by bi-Lipschitz map.

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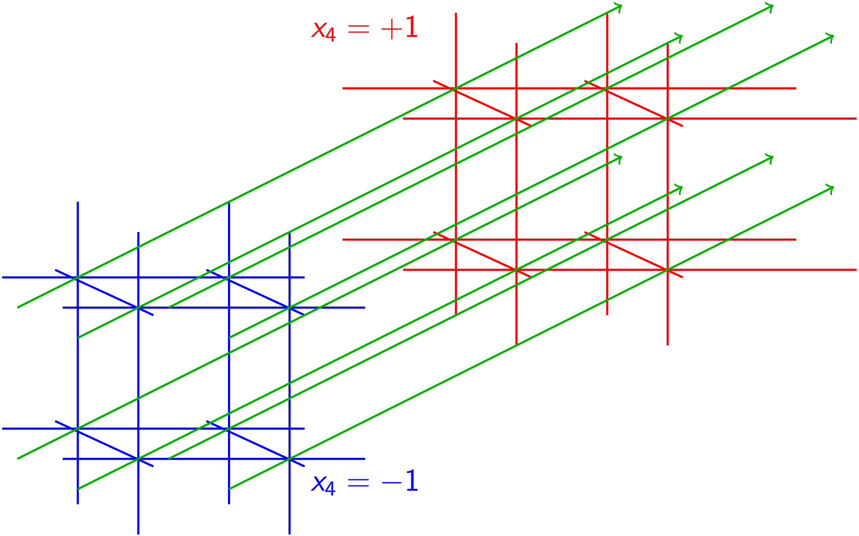
'Reinflate' the Cantor set back up to one that has positive measure.

Something crazy



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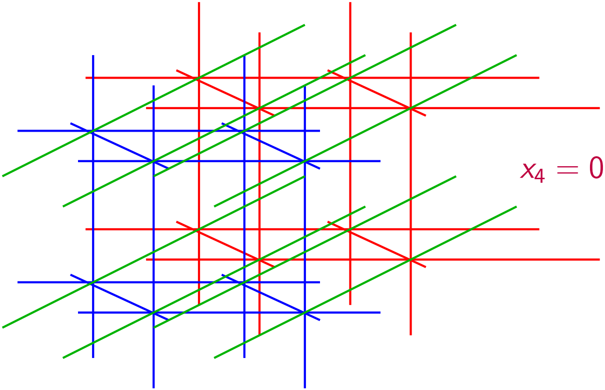

$x_4 = 0$

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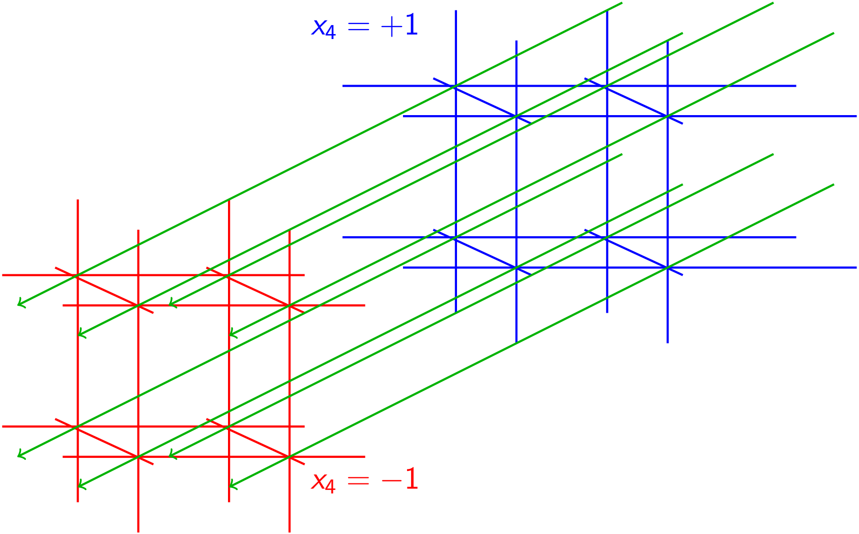

A Cantor Tower